Oscilador Armónico IV

D-1. Mean values and root mean square deviations of X and P in a state $|\varphi_n\rangle$

Neither X nor P commutes with H, and the eigenstates $|\varphi_n\rangle$ of H are not eigenstates of X or P. Consequently, if the harmonic oscillator is in a stationary state $|\varphi_n\rangle$, a measurement of the observable X or the observable P can, *a priori*, yield any result (since the spectra of X and P include all real numbers). We shall now calculate the mean values of X and P in such a stationary state and then their root mean square deviations ΔX and ΔP, which will enable us to verify the uncertainty relation.

As we indicated in § C-1-c, we shall perform these calculations with the help of the operators a and $a^\dagger$. As far as the mean values of X and P are concerned, the result follows directly from formulas (C-22), which show that neither X nor P has diagonal matrix elements:

$$\begin{aligned}\langle\varphi_n|X|\varphi_n\rangle &= 0\\ \langle\varphi_n|P|\varphi_n\rangle &= 0\end{aligned} \tag{D-1}$$

To obtain the root mean square deviations ΔX and ΔP, we must calculate the mean values of X^2 and P^2:

$$\begin{aligned}(\Delta X)^2 &= \langle\varphi_n|X^2|\varphi_n\rangle - (\langle\varphi_n|X|\varphi_n\rangle)^2 = \langle\varphi_n|X^2|\varphi_n\rangle\\ (\Delta P)^2 &= \langle\varphi_n|P^2|\varphi_n\rangle - (\langle\varphi_n|P|\varphi_n\rangle)^2 = \langle\varphi_n|P^2|\varphi_n\rangle\end{aligned} \tag{D-2}$$

Now, according to (B-1) and (B-7):

$$\begin{aligned}X^2 &= \frac{\hbar}{2m\omega}(a^\dagger + a)(a^\dagger + a)\\ &= \frac{\hbar}{2m\omega}(a^{\dagger 2} + aa^\dagger + a^\dagger a + a^2)\\ P^2 &= -\frac{m\hbar\omega}{2}(a^\dagger - a)(a^\dagger - a)\\ &= -\frac{m\hbar\omega}{2}(a^{\dagger 2} - aa^\dagger - a^\dagger a + a^2)\end{aligned} \tag{D-3}$$

The terms in a^2 and $a^{\dagger 2}$ do not contribute to the diagonal matrix elements, since $a^2|\varphi_n\rangle$ is proportional to $|\varphi_{n-2}\rangle$, and $a^{\dagger 2}|\varphi_n\rangle$ to $|\varphi_{n+2}\rangle$; both are orthogonal to $|\varphi_n\rangle$. On the other hand:

$$\begin{aligned}\langle\varphi_n|(a^\dagger a + aa^\dagger)|\varphi_n\rangle &= \langle\varphi_n|(2a^\dagger a + 1)|\varphi_n\rangle\\ &= 2n + 1\end{aligned} \tag{D-4}$$

Consequently:

$$(\Delta X)^2 = \langle\varphi_n|X^2|\varphi_n\rangle = \left(n + \frac{1}{2}\right)\frac{\hbar}{m\omega} \tag{D-5a}$$

$$(\Delta P)^2 = \langle\varphi_n|P^2|\varphi_n\rangle = \left(n + \frac{1}{2}\right)m\hbar\omega \tag{D-5b}$$

Propiedades de $|\phi_0\rangle$

- $E \neq 0$ a diferencia del caso clásico.
- Tiene extensión espacial
- Balance T y U + principio de incertidumbre

Evolucion temporal

D-3. Time evolution of the mean values

Consider a harmonic oscillator whose state at $t = 0$ is:

$$|\psi(0)\rangle = \sum_{n=0}^{\infty} c_n(0)|\varphi_n\rangle \qquad \text{(D-22)}$$

($|\psi(0)\rangle$ is assumed to be normalized). Its state $|\psi(t)\rangle$ at t can be obtained by using rule (D-54) of Chapter III:

$$\begin{aligned}|\psi(t)\rangle &= \sum_{n=0}^{\infty} c_n(0)\,\mathrm{e}^{-i\,E_n t/\hbar}|\varphi_n\rangle \\ &= \sum_{n=0}^{\infty} c_n(0)\,\mathrm{e}^{-i\left(n+\frac{1}{2}\right)\omega t}|\varphi_n\rangle\end{aligned} \qquad \text{(D-23)}$$

The mean value of any physical quantity A is therefore given as a function of time by:

$$\langle\psi(t)|A|\psi(t)\rangle = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_m^*(0)\; c_n(0)\; A_{mn}\,\mathrm{e}^{i(m-n)\omega t} \qquad \text{(D-24)}$$

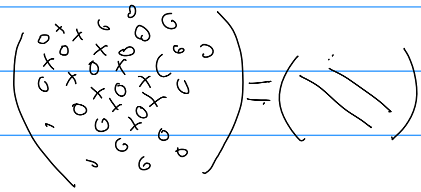
with:

$$A_{mn} = \langle\varphi_m|A|\varphi_n\rangle \qquad \text{(D-25)}$$

frecuencia ω y armóicos

$A = X$

$$\langle\varphi_m|X|\varphi_n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n+1}\,\delta_{m,n+1} + \sqrt{n}\,\delta_{m,n-1}\right]$$



No quedan los armónicos

P también tiene una forma similar

$\therefore$ En $\langle P\rangle(t)$ y $\langle X\rangle(t)$ sólo aparecen términos que oscilan como ωt.

Ejemplos de evolución temporal

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty} C_n(0)\,e^{-i\omega(n+\frac{1}{2})t}|\varphi_n\rangle$$

$\rightarrow$ Si $C_{n'}(0) = 1$ y los demás $C_n(0) = 0$

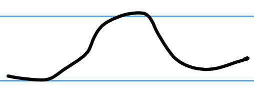
$$|\Psi(t)\rangle = e^{-i\omega(n'+\frac{1}{2})t}|\varphi_{n'}\rangle$$

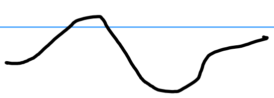
$$\Psi(x,t) = e^{-i\omega(n'+\frac{1}{2})t}\,\varphi_{n'}(x)$$

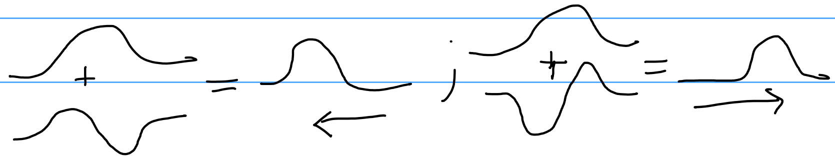
$\therefore$ Si el estado inicial es un e-estado de $\hat{H}$, no cambia la forma de $\Psi(x,t)$ con el tiempo.

$\rightarrow$ $C_0(0) = \frac{1}{\sqrt{2}}$, $C_1(0) = \frac{1}{\sqrt{2}}$, los demás son cero.

$$|\varphi(t)\rangle = \frac{e^{\frac{-i\omega t}{2}}|\varphi_0\rangle + e^{\frac{-i3\omega t}{2}}|\varphi_1\rangle}{\sqrt{2}}$$

$\varphi_0 =$ 

$\varphi_1 =$ 



Estados coherentes del oscilador armónico

- Vimos qe $\langle \phi_n | X | \phi_n \rangle = \langle \phi_n | P | \phi_n \rangle = 0$

- Clásicamente x y p oscilan y $x = p = 0$ sólo si $E = 0$.

- Para $E \gg \hbar\omega$ esperaríamos comportamiento clásico

- ¿Es posible construir un estado cuántico cuyas predicciones sean iguales al caso clásico?
- Veremos que sí: estados coherentes ó casi clásicos.

- Estados de Glauber (1963)

- Modelan el estado de la luz que sale de un láser.

- $[X, P] \neq 0$
$\Rightarrow$ No podemos construir un estado en el que ambos estén perfectamente bien def.

The Nobel Prize in Physics 2005

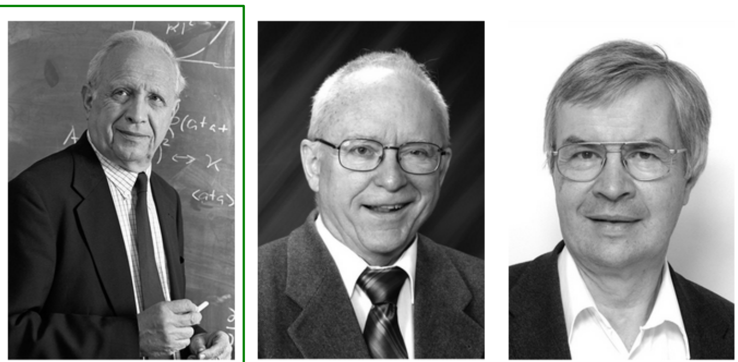


Photo: J.Reed
Roy J. Glauber
Prize share: 1/2

Photo: Sears.P.Studio
John L. Hall
Prize share: 1/4

Photo: F.M. Schmidt
Theodor W. Hänsch
Prize share: 1/4

The Nobel Prize in Physics 2005 was divided, one half awarded to Roy J. Glauber "for his contribution to the quantum theory of optical coherence", the other half jointly to John L. Hall and Theodor W. Hänsch "for their contributions to the development of laser-based precision spectroscopy, including the optical frequency comb technique."

Buscamos un $|\psi\rangle$ para el que $\langle x \rangle$, $\langle p \rangle$ y $\langle H \rangle$ sean muy parecidos a sus valores clásicos.