Consecuencias de la ecuación de Schrödinger

- $i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$

- Con dar $|\psi(t_0)\rangle$ podemos obtener $|\psi(t)\rangle$ $\forall t$

Se puede dar como condición inicial cualquiera de estas

- $\langle \vec{p}|\psi(t_0)\rangle = \psi(\vec{p}, t_0)$ $\quad$ $\langle \vec{r}|\psi(t_0)\rangle = \psi(\vec{r}, t_0)$

  o $|\psi(t_0)\rangle$ directamente.

- La podemos escribir como $\left(i\hbar \frac{d}{dt} - H\right)|\psi(t)\rangle = 0$

- Si tenemos dos soluciones $|\psi_1(t)\rangle$, $|\psi_2(t)\rangle$

$\left(i\hbar \frac{d}{dt} - H\right)\lambda_1|\psi_1(t)\rangle = 0$; $\left(i\hbar \frac{d}{dt} - H\right)\lambda_2|\psi_2(t)\rangle = 0$

$$\Rightarrow \quad \left(i\hbar \frac{d}{dt} - H\right)\left(\lambda_1|\psi_1(t)\rangle + \lambda_2|\psi_2(t)\rangle\right) = 0$$

$\therefore$ $\lambda_1|\psi_1(t)\rangle + \lambda_2|\psi_2(t)\rangle$ es solución también

$\Rightarrow$ Podemos hablar de un operador de evolución temporal

$$|\psi(t)\rangle = U(t, t_0)\,|\psi(t_0)\rangle$$

- Podemos empezar a sospechar que $U(t, t_0) = e^{-\frac{i}{\hbar}\overset{\text{operador}}{H}(t-t_0)}$

Conservación de la probabilidad

Si $\langle\psi(t_0)|\psi(t_0)\rangle = 1$, ¿Pasará que $\langle\psi(t)|\psi(t)\rangle = 1$?

Queremos ver si $\frac{d}{dt}\langle\psi(t)|\psi(t)\rangle = 0$

$$\frac{d}{dt}\langle\psi(t)|\psi(t)\rangle = \left[\frac{d}{dt}\langle\psi(t)|\right]|\psi(t)\rangle + \langle\psi(t)|\left[\frac{d}{dt}|\psi(t)\rangle\right]$$

$\frac{d}{dt}|\psi(t)\rangle = \frac{1}{i\hbar}H|\psi(t)\rangle$

$\frac{d}{dt}\langle\psi(t)| = -\frac{1}{i\hbar}\langle\psi(t)|H$

$$= -\frac{1}{i\hbar}\cancel{\langle\psi(t)|H|\psi(t)\rangle} + \frac{1}{i\hbar}\cancel{\langle\psi(t)|H|\psi(t)\rangle} = 0$$

$$\therefore \langle\psi(t)|\psi(t)\rangle = \text{constante.}$$

En la representación de posición

$$\Rightarrow \boxed{\int|\psi(\vec{r},t)|^2 d^3r = 1 \quad \forall t} \left(\text{Suponiendo } \int|\psi(\vec{r},t_0)|^2 d^3r = 1\right)$$

Esto es una ley de conservación global.

¿Habrá una local como $\frac{\partial}{\partial t}\rho(\vec{r},t) + \nabla\cdot\vec{J}(\vec{r},t) = 0$ como en el caso de cargas?

Si definimos $\rho(\vec{r},t) = |\psi(\vec{r},t)|^2 = \psi(\vec{r},t)\psi^*(\vec{r},t)$

$$\frac{\partial}{\partial t}\rho(\vec{r},t) = \frac{\partial}{\partial t}\left(\psi(\vec{r},t)\psi^*(\vec{r},t)\right) = \left(\frac{\partial}{\partial t}\psi(\vec{r},t)\right)\psi^*(\vec{r},t) + \psi(\vec{r},t)\left(\frac{\partial}{\partial t}\psi^*(\vec{r},t)\right)$$

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{r},t) = \left(-\frac{\hbar^2\nabla^2}{2m} + V(\vec{r})\right)\psi(\vec{r},t)$$

$$-i\hbar\frac{\partial}{\partial t}\psi^*(\vec{r},t) = \left(-\frac{\hbar^2\nabla^2}{2m} + V(\vec{r})\right)\psi^*(\vec{r},t)$$

$$= \left[\frac{1}{i\hbar}\left(-\frac{\hbar^2\nabla^2}{2m} + \cancel{V(\vec{r})}\right)\psi\right]\psi^* - \psi\left[\frac{1}{i\hbar}\left(-\frac{\hbar^2\nabla^2}{2m} + \cancel{V(\vec{r})}\right)\psi^*\right]$$

$$= \frac{\hbar}{2mi}\left[-(\nabla^2\psi)\psi^* + \psi(\nabla^2\psi^*)\right]$$

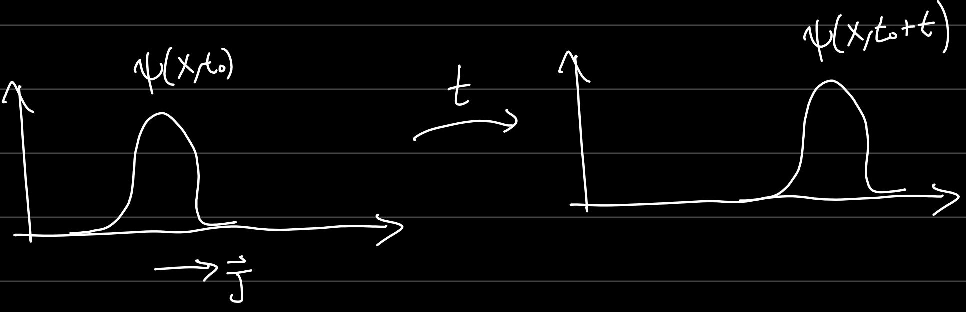
$$= \frac{\hbar}{2mi}\left[\psi\nabla^2\psi^* - \psi^*\nabla^2\psi\right]$$

Si definimos

$$\vec{J}(\vec{r},t) = \frac{\hbar}{2mi}\left[\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right]$$ Corriente de probabilidad

$$\nabla\cdot\vec{J}(\vec{r},t) = \frac{\hbar}{2mi}\left[\cancel{(\vec{\nabla}\psi^*)\cdot(\vec{\nabla}\psi)} + \psi^*\nabla^2\psi - \cancel{(\vec{\nabla}\psi)\cdot(\vec{\nabla}\psi^*)} - \psi(\nabla^2\psi^*)\right]$$

$$\therefore \quad \boxed{\frac{\partial}{\partial t}\rho(\vec{r},t) + \nabla\cdot\vec{J}(\vec{r},t) = 0}$$



Ejemplo

Partícula libre $\quad H = \frac{p^2}{2m}$

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{r},t) = -\frac{\hbar^2\nabla^2}{2m}\psi(\vec{r},t) \rightarrow \psi(\vec{r},t) = Ae^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\rho = |\psi(\vec{r},t)|^2 = |A|^2$$

$$\vec{J} = \frac{\hbar}{2m\cancel{i}}\left[A^*\cancel{e^{-i(\vec{k}\cdot\vec{r}-\omega t)}}\,A\,\cancel{i}\vec{k}\,\cancel{e^{i(\vec{k}\cdot\vec{r}-\omega t)}} - A\cancel{e^{i(\vec{k}\cdot\vec{r}-\omega t)}}\,A^*(-\cancel{i}\vec{k})\,\cancel{e^{-i(\vec{k}\cdot\vec{r}-\omega t)}}\right]$$

$$= \frac{\hbar\vec{k}}{2m}\left[2|A|^2\right]$$

Evolución temporal del valor promedio de un observable.

- La evolución temporal de un valor esperado está dada por

$$\langle A\rangle(t) = \langle\psi(t)|\,A\,|\psi(t)\rangle \quad \ldots \quad (1)$$

- En general, el operador A podría depender explícitamente del tiempo.

- Buscamos una ecuación de evolución para $\langle A\rangle(t)$

- Derivando (1) respecto a t

$$\frac{d}{dt}\langle A\rangle(t)=\frac{d}{dt}\langle\psi(t)|A(t)|\psi(t)\rangle=\left[\frac{d}{dt}\langle\psi(t)|\right]A(t)|\psi(t)\rangle+\langle\psi(t)|\frac{\partial A}{\partial t}(t)|\psi(t)\rangle+\langle\psi(t)|A(t)\left[\frac{d}{dt}|\psi(t)\rangle\right]$$

$\frac{d}{dt}|\psi(t)\rangle=\frac{1}{i\hbar}H|\psi(t)\rangle$

$\frac{d}{dt}\langle\psi(t)|=-\frac{1}{i\hbar}\langle\psi(t)|H$

$$=\frac{1}{i\hbar}\left[\langle\psi(t)|\underline{AH}|\psi(t)\rangle-\langle\psi(t)|\underline{HA}|\psi(t)\rangle\right]+\left\langle\frac{\partial A}{\partial t}\right\rangle$$

$$\frac{d}{dt}\langle A\rangle=\frac{1}{i\hbar}\langle[A,H]\rangle+\left\langle\frac{\partial A}{\partial t}\right\rangle$$

Caso particular: los observables $\vec{R}$ y $\vec{P}$ para una partícula en un potencial (Teorema de Ehrenfest)

$$H=\frac{P^2}{2m}+V(\vec{R})$$

En este caso $\frac{\partial\vec{R}}{\partial t}=0$ y $\frac{\partial\vec{P}}{\partial t}=0$.

$$\frac{d}{dt}\langle\vec{R}\rangle=\frac{1}{i\hbar}\langle[\vec{R},H]\rangle=\frac{1}{i\hbar}\left\langle\left[\vec{R},\frac{\vec{P}^2}{2m}\right]\right\rangle=\frac{1}{m}\langle\vec{P}\rangle$$

$$\left[\vec{R},\frac{P^2}{2m}+V(\vec{R})\right]=\left[\vec{R},\frac{P^2}{2m}\right]+[\vec{R},\cancel{V(\vec{R})}]^{0}$$

Recordando que $[\underset{R}{A},\underset{P}{B}\underset{P}{C}]=[A,B]C+B[A,C]$

$$\left[\vec{R},\frac{P^2}{2m}\right]=\frac{i\hbar}{m}\vec{P}$$ (usando $[R_i,P_j]=i\hbar\delta_{ij}$

$$\frac{d}{dt}\langle\vec{P}\rangle=\frac{1}{i\hbar}\langle[\vec{P},H]\rangle=\frac{1}{i\hbar}\langle[\vec{P},V(\vec{R})]\rangle=-\langle\nabla V(R)\rangle$$

tarea

Obtuvimos

Paul Ehrenfest

$$\frac{d}{dt}\langle\vec{R}\rangle=\frac{1}{m}\langle\vec{P}\rangle$$
$$\frac{d}{dt}\langle\vec{P}\rangle=-\langle\nabla V(\vec{R})\rangle$$

derivando la de arriba y enchufando la de abajo $\Longrightarrow$

$$m\frac{d^2}{dt^2}\langle\vec{R}\rangle=-\langle\nabla V(\vec{R})\rangle$$

2a ley de Newton.