Momento angular II

$\vec{L} = \vec{R} \times \vec{P}$ (orbital)

$\vec{S}$ (espín)

$\vec{J}$ (momento angular general)

$[J_x, J_y] = i\hbar J_z$, $[J_z, J_x] = i\hbar J_y$, $[J_y, J_z] = i\hbar J_x$

$$J^2 = J_x^2 + J_y^2 + J_z^2$$

$[\vec{J}^2, J_i] = 0$ para $i = x, y, z$

Escogemos J_z para encontrar una base de e.V comunes con J^2

$$J_\pm = J_x \pm i J_y$$

Los eV de J^2, J_z tienen la forma

$$|k, j, m\rangle$$

índice de eV J^2

índice para J_z

$$J^2 |k,j,m\rangle = \hbar^2 j(j+1) \, |k,j,m\rangle$$
$$J_z |k,j,m\rangle = \hbar m \, |k,j,m\rangle$$

Vimos que $j \geq 0$; Sus valores posibles son

$$j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \tfrac{5}{2}, \ldots$$

Dado j, los valores posibles de m son

$$-j \leq m \leq j$$

$$m = -j, -j+1, \dots, j-1, j$$

$\Rightarrow$ j es entero $\Leftrightarrow$ m es entero

j es semi-entero $\Leftrightarrow$ m es semi-entero

$$|k, j, m\rangle$$

asociado a la magnitud del m.a.

asociado a la componente z de m.a.

$$\langle k, j, m | J^2 | k, j, m \rangle = \hbar^2 j(j+1)$$

$$\langle k, j, m | \sqrt{J^2} | k, j, m \rangle = \hbar \sqrt{j(j+1)}$$

$$A|a\rangle = a|a\rangle \Rightarrow F(A)|a\rangle = F(a)|a\rangle$$

Partiendo de un e.V. J^2, J_z $|k, j, m\rangle$ podemos encontrar más e.V usando J_+ y J_-

$$J_\pm |k, j, m\rangle = C\,|k, j, m\pm 1\rangle$$

$$\langle k, j, m | J_\mp J_\pm | k, j, m \rangle = \langle k, j, m | J^2 - J_z^2 \mp \hbar J_z | k, j, m \rangle$$

$$= \cancel{\langle k, j, m | k, j, m \rangle}^{\,1} \left(\hbar^2 j(j+1) - \hbar^2 m^2 \mp \hbar m\right)$$

$$= \hbar^2 [j(j+1) - m(m \pm 1)] = C^2$$

$$\boxed{\begin{gathered} J_\pm|k,j,m\rangle = \hbar\sqrt{j(j+1)-m(m\pm1)}\;|k,j,m\pm1\rangle \\ J^2|k,j,m\rangle=\hbar^2 j(j+1)\,|k,j,m\rangle \\ J_z|k,j,m\rangle=\hbar m\,|k,j,m\rangle \end{gathered}}$$

- Los valores que toma j dependerá del problema en particular.

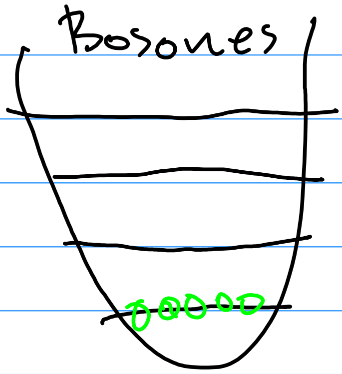
Para momento angular orbital j es entero

Para e^-, protones, neutrones, j es semi-entero

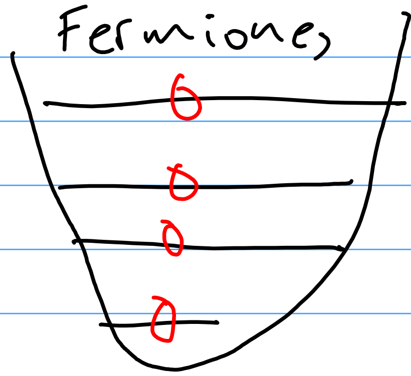
Fermiones tienen espín semi-entero

Bosones " " entero.

Bosones



Fermiones



$p\ s=\frac{1}{2}$ $\quad e\ s=\frac{1}{2}$ $\xrightarrow{+}$ Átomo $s = 1, 0$

$j=1/2 \quad m = +1/2, -\frac{1}{2}$

Estructura del espacio vectorial

- Para k y j fijos hay $2j+1$ vectores ortonormales $|k,j,m\rangle$ correspondientes a $m=-j,\dots,j$

- En general

$$\langle k',j',m'|k,j,m\rangle = \delta_{jj'}\delta_{mm'}\delta_{kk'}$$

- Para construir la base tomar $|k,j,m=j\rangle$ y usando J_- obtener los vectores para otras m.

$$\sum_k \sum_j \sum_{m=-j}^{j} |k,j,m\rangle\langle k,j,m| = \mathbb{1}$$

Representación matricial de $J^2, J_z, J_+, J_-, J_y, J_x$

$$\langle k',j',m'|J^2|k,j,m\rangle = \hbar^2 j(j+1)\,\delta_{k'k}\delta_{j'j}\delta_{m'm}$$

no depende de k ni de m.

$$\langle k',j',m'|J_z|k,j,m\rangle = \hbar m\,\delta_{k'k}\delta_{j'j}\delta_{m'm}$$

$$\langle k',j',m'|J_\pm|k,j,m\rangle = \hbar\sqrt{j(j+1)-m(m\pm1)}\,\langle k',j',m'|k,j,m\pm1\rangle$$

$$= \hbar\sqrt{j(j+1)-m(m\pm1)}\,\delta_{k'k}\delta_{j'j}\delta_{m',m\pm1}$$

$$J_+ = J_x + iJ_y \qquad J_- = J_x - iJ_y$$

$$J_x = \frac{J_+ + J_-}{2} \qquad J_y = \frac{J_+ - J_-}{2i}$$

(Ojo J_+, J_-, J_x, J_y no son diagonales)

Ejemplos

(Ignoraremos κ)

i) $j=0 \longrightarrow m=0 \longrightarrow$ Base: $\{|j=0, m=0\rangle\}$

espacio de dim $=1$

$$\left(J^2\right)^{(0)} = 0 \qquad \left(J_z\right)^{(0)} = 0$$

ii) $j=\frac{1}{2} \longrightarrow m=\frac{1}{2},-\frac{1}{2} \longrightarrow$ Base: $\{|j=\frac{1}{2}, m=\frac{1}{2}\rangle, |j=\frac{1}{2}, m=-\frac{1}{2}\rangle\}$

$$\underline{\left(J_z\right)^{(1/2)} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

$$\left.\begin{array}{l} \left(J_+\right)^{(1/2)} = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \left(J_-\right)^{(1/2)} = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{array}\right\} \Longrightarrow \begin{array}{l} \underline{\left(J_x\right)^{(\frac{1}{2})} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \\ \underline{\left(J_y\right)^{(\frac{1}{2})} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}} \end{array}$$

Matrices de Pauli S_x, S_y, S_z

iii) $j=1 \Rightarrow m=-1,0,1 \rightarrow$ Base $\{|1\,1\rangle, |1,0\rangle, |1,-1\rangle\}$

$\uparrow\uparrow$ $j\,m$

$$\left(J_z\right)^{(1)} = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\left.\begin{array}{l} \left(J_+\right)^{(1)} = \hbar\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \\ \left(J_-\right)^{(1)} = \hbar\begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \end{array}\right\} \quad \begin{array}{l} \left(J_x\right)^{(1)} = \ldots \\ \left(J_y\right)^{(1)} = \ldots \end{array}$$

iv) Si k pudiera tomar los valores $\{0, 1, 2, \ldots\}$

y $j = 1/2$

Para $|k, j, m\rangle : \{|0, \frac{1}{2}, \frac{1}{2}\rangle, |0, \frac{1}{2}, -\frac{1}{2}\rangle, |1, \frac{1}{2}, \frac{1}{2}\rangle, |1, \frac{1}{2}, -\frac{1}{2}\rangle, \ldots\}$

$$J_z = \hbar \begin{pmatrix} \underbrace{\begin{matrix} 1/2 & 0 \\ 0 & -\frac{1}{2} \end{matrix}}_{k=0} & & & \\ & \underbrace{\begin{matrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{matrix}}_{k=1} & & \\ & & \underbrace{\begin{matrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{matrix}}_{k=2} & \\ & & & \ddots \end{pmatrix}$$

Momento angular orbital $\vec{L}$

- Regresamos a $\vec{L} = \vec{R} \times \vec{P}$
- Describirlo en $\{|\vec{r}\rangle\}$

$\rightarrow$ $\vec{L}^2$, L_z Buscamos e.V y e.v.

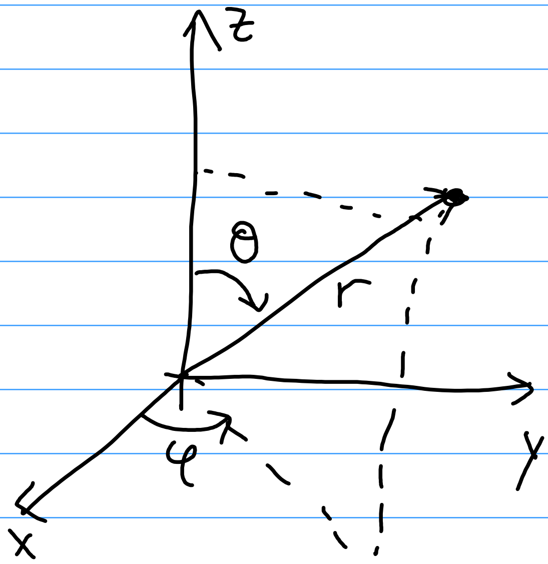
Aplicar $\vec{R}$ es multiplicar por $\vec{r}$
" $\vec{P}$ es aplicar $-i\hbar\nabla$

$$L_x = Y P_z - Z P_y = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$L_x\psi(\vec{r}) = \langle \vec{r}|L_x|\psi\rangle = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\psi(\vec{r})$$

Vamos a usar coordenadas esféricas

$$(r, \theta, \varphi)$$



$$d^3r = r^2 \sin\theta \, d\theta \, d\varphi$$
$$= r^2 dr \, d\Omega$$

$$d\Omega = \sin\theta \, d\theta \, d\varphi$$

$$L_x = i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{\tan\theta}\frac{\partial}{\partial\varphi}\right)$$

$$L_y = i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \frac{\sin\varphi}{\tan\theta}\frac{\partial}{\partial\varphi}\right)$$

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\varphi}$$

$$L^2 = -\hbar^2 \left(\frac{\partial^2}{\partial^2\theta} + \frac{1}{\tan\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \right)$$

$$L_+ = \hbar e^{i\varphi} \left(\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\varphi} \right)$$

$$L_- = \hbar e^{-i\varphi} \left(-\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\varphi} \right)$$

Para un $|\psi\rangle$ arbitrario

$$L^2 |\psi\rangle = \hbar^2 \ell(\ell+1) |\psi\rangle$$

$$L_z |\psi\rangle = \hbar m |\psi\rangle$$

$$\langle \vec{r} | L^2 | \psi \rangle = \hbar^2 \ell(\ell+1) \langle \vec{r} | \psi \rangle = \hbar^2 \ell(\ell+1) \psi(\vec{r})$$

$$\langle \vec{r} | L_z | \psi \rangle = \hbar m \psi(\vec{r})$$

$$-\hbar^2 \left(\frac{\partial^2}{\partial^2\theta} + \frac{1}{\tan\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \right) \psi(\vec{r}) = \hbar^2 \ell(\ell+1) \psi(\vec{r})$$

$$\frac{\hbar}{i} \frac{\partial}{\partial\varphi} \psi(\vec{r}) = \hbar m \psi(\vec{r})$$

No aparece r, si hacemos separación de variables

$$\psi(r, \theta, \varphi) = R(r) Y(\theta, \varphi)$$

También sabemos que $Y_\ell^m(\theta, \varphi)$

$$-i \frac{\partial}{\partial\varphi} Y_\ell^m(\theta, \varphi) = m Y_\ell^m(\theta, \varphi)$$

Sep. de var. $\quad Y_\ell^m(\theta, \varphi) = F_\ell^m(\theta) G_\ell^m(\varphi)$

$$-i\, G_l^{m\prime}(\varphi) = m\, G_l^m(\varphi) \Rightarrow G_l^m(\varphi) = e^{im\varphi}$$

$$Y_l^m(\theta,\varphi) = F_l^m(\theta)\, e^{im\varphi}$$

$$Y_l^m(\theta,\varphi) = Y_l^m(\theta,\varphi+2\pi)$$

$$\cancel{F_l^m(\theta)}\, \cancel{e^{im\varphi}} = \cancel{F_l^m(\theta)}\, e^{im(\cancel{\varphi}+2\pi)}$$ dividiendo

$$e^{2im\pi} = 1 \Rightarrow m \text{ es entero}$$
$$\Rightarrow l \text{ es entero.}$$

$\therefore$ Para momento angular orbital l y m deben ser enteros.

Para encontrar $F_l^m(\theta)$ recordemos que:

$$L_+ Y_l^{m=l} = 0 \quad ; \quad L_- Y_l^{m=-l} = 0$$

$$\hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\varphi}\right) Y_l^l(\theta,\varphi) = 0$$

$$\left(\frac{\partial}{\partial\theta} + i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right) F_l^l(\theta)\, e^{il\varphi} = 0$$

$$\left(\frac{\partial}{\partial\theta} - l\frac{\cos\theta}{\sin\theta}\right) F_l^l(\theta) = 0$$

$$\frac{1}{F(\theta)}\frac{dF}{d\theta}(\theta) = \frac{l\cos\theta}{\sin\theta} = \frac{f'(\theta)}{f(\theta)}$$

$$\ln F(\theta) = l\ln(\sin\theta) = \ln f(\theta)$$

$$F_l^l(\theta) = \sin^l(\theta)$$

$$Y_\ell^\ell(\theta,\varphi) = \sin^\ell(\theta)\, e^{i\ell\varphi}\, N$$

cte de normalización

N se determina por

$$\int_0^{2\pi}\int_0^{\pi} |Y_\ell^\ell(\theta,\varphi)|^2 \sin\theta \, d\theta \, d\varphi = 1$$

Aplicando L_- obtenemos las otras $Y_\ell^m(\theta,\varphi)$

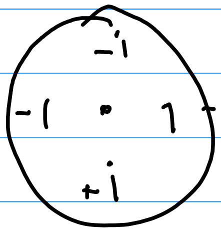
$l=0$	$l=1$	$l=2$
$Y_0^0 = \sqrt{\frac{1}{4\pi}}$	$Y_1^{\pm1} = \pm\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\varphi}$ $Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$	$Y_2^{\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\varphi}$ $Y_2^{\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\varphi}$ $Y_2^0 = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$

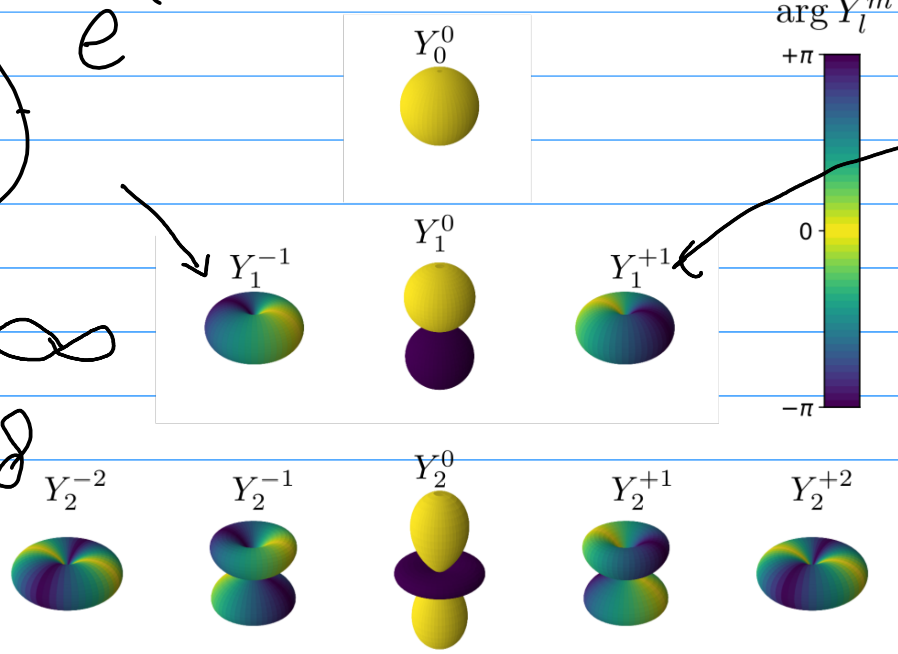
La expresión general para cualquier valor de l y m está dada por

$$Y_l^m(\theta,\varphi) = (-1)^m\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\varphi}, \quad m \geq 0;$$
$$Y_l^{-m}(\theta,\varphi) = (-1)^m Y_l^m(\theta,\varphi)^*;$$
$$P_l^m(x) = (1-x^2)^{|m/2|}\frac{d^{|m|}}{dx^{|m|}}P_l(x);$$
$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l.$$

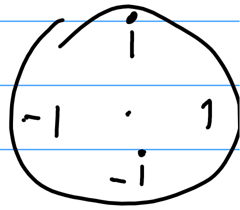
Aquí, a $P_l^m(x)$ se le conoce como polinomio asociado de Legendre.

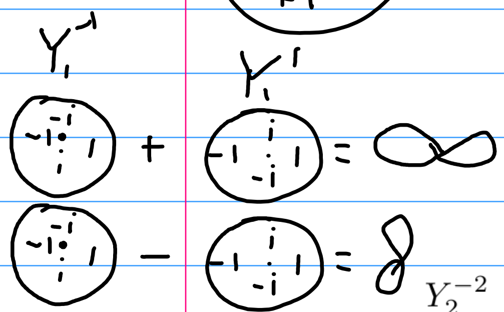
$e^{-i\varphi}$





$e^{i\varphi}$





Interpretación

Si tenemos un eigenestado $|l,m\rangle$

Si medimos L^2 obtendremos $\hbar^2 l(l+1)$

Si medimos L_z " $\hbar m$

Si medimos L_x o L_y no podemos predecir el resultado.

Podemos predecir el promedio de las mediciones

$$\langle l,m| L_x | l,m\rangle = \langle l,m| \frac{L_+ + L_-}{2} |l,m\rangle$$

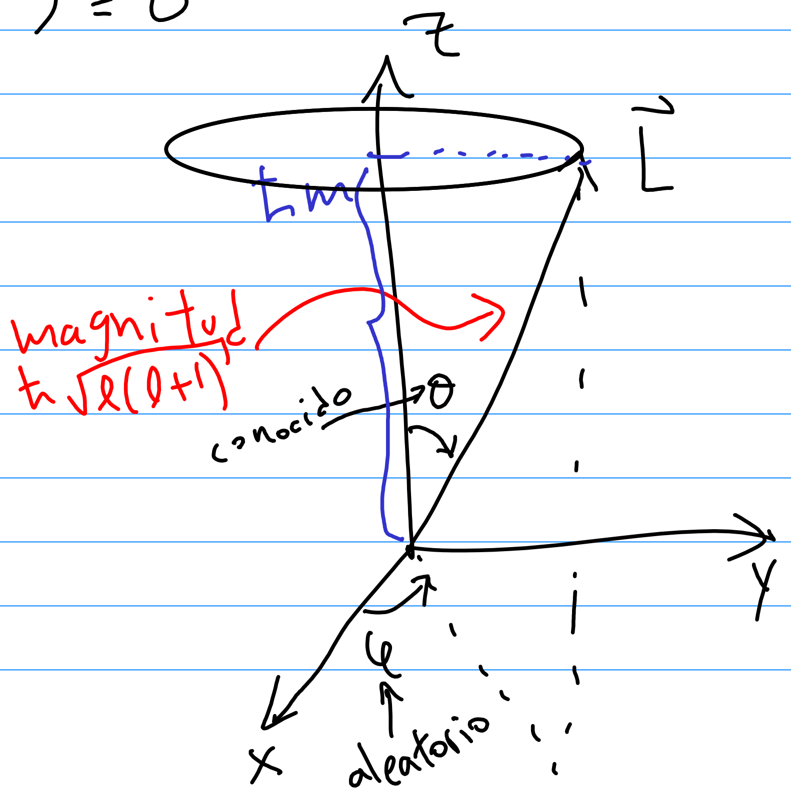
$$= \langle lm | (\sqrt{\cdots}\, |l,m+1\rangle + \sqrt{\cdots}\, |l,m-1\rangle)$$

$$= 0$$

$$\langle l,m| L_y | l,m\rangle = 0$$

$$\langle L_x\rangle = 0$$

$$\langle L_x^2\rangle$$



$$\cos\theta = \frac{m}{\sqrt{l(l+1)}}$$

$$\cos\theta_{min} = \frac{l}{\sqrt{l^2+l}} < 1$$

$$\theta_{min} > 0$$