Momento angular cuántico V

$$\vec{L} = \vec{R}\times\vec{P} \longrightarrow [J_i, J_j] = i\hbar\,\epsilon_{ijk} J_k$$

$$\begin{cases} J^2 |j\, m\rangle = \hbar^2 j(j+1)\,|j\, m\rangle \\ J_z |j\, m\rangle = \hbar m\,|j\, m\rangle \end{cases}$$

$$\vec{L} = \vec{R}\times\vec{P}$$

$$\langle \vec{r}\,|\vec{R}|\,\psi\rangle = \vec{r}\,\psi(x)$$
$$\langle \vec{r}\,|\vec{P}|\,\psi\rangle = -i\hbar\vec{\nabla}\psi(x)$$

$$\begin{cases} -\left\{\dfrac{\partial^2}{\partial\theta^2} + \dfrac{1}{\tan\theta}\dfrac{\partial}{\partial\theta} + \dfrac{1}{\sin^2\theta}\dfrac{\partial^2}{\partial\varphi^2}\right\}\psi(r,\theta,\varphi) = l\,(\,l+1)\,\psi(r,\theta,\varphi) & \text{(D-7-a)} \\ -\,i\dfrac{\partial}{\partial\varphi}\,\psi(r,\theta,\varphi) = m\,\psi(r,\theta,\varphi) & \text{(D-7-b)} \end{cases}$$

$$L^2\,\psi(r,\theta,\varphi) = \hbar^2\, l(l+1)\,\psi(r,\theta,\varphi)$$
$$L_z\,\psi(r,\theta,\varphi) = \hbar m\,\psi(r,\theta,\varphi)$$

$$\Downarrow$$

$$\psi(r,\theta,\varphi) = f(r)\,Y_l^m(\theta,\varphi)$$

$$L^2\,Y_l^m(\theta,\varphi) = \hbar^2\, l(l+1)\,Y_l^m(\theta,\varphi)$$
$$L_z\,Y_l^m(\theta,\varphi) = \hbar m\,Y_l^m(\theta,\varphi)$$

$$\Downarrow$$

$$Y_l^m(\theta,\varphi) = F_l^m(\theta)\,e^{im\varphi}$$

entero

$\therefore$ l entero

$$Y_l^m(\theta,\varphi) = Y_l^m(\theta,\varphi+2\pi)$$

- Método algebráico para encontrar $Y_l^m(\theta,\varphi)$
  - $\rightarrow$ $L_+ Y_l^{m=l}(\theta,\varphi) = 0$ resolver para $Y_l^l(\theta,\varphi) \leftarrow$
  - $\rightarrow$ Aplicando L_- varias veces obtener $Y_l^m(\theta,\varphi)$

$$L_\pm |l, m\rangle = \hbar\sqrt{l(l+1) - m(m\pm1)}\;|l, m\pm1\rangle$$

L_+ L_+ L_+

$$m = -l,\ -l+1,\ \ldots,\ l-1,\ l \qquad \times$$

L_- L_- L_-

$\vec{L} = \vec{R} \times \vec{P}$ $\qquad \langle \vec{r} | \vec{L} | \psi \rangle = -i\hbar\, \vec{r} \times \vec{\nabla}\, \psi(\vec{r}) \Rightarrow \vec{L} \sim \frac{\hbar}{i} \vec{r} \times \vec{\nabla}$

$$L_x = \frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$
$$L_y = \frac{\hbar}{i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$
$$L_z = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$\xrightarrow{\text{en esféricos}}$

$$L_x = i\hbar\left(\sin\varphi \frac{\partial}{\partial\theta} + \frac{\cos\varphi}{\tan\theta}\frac{\partial}{\partial\varphi}\right)$$
$$L_y = i\hbar\left(-\cos\varphi \frac{\partial}{\partial\theta} + \frac{\sin\varphi}{\tan\theta}\frac{\partial}{\partial\varphi}\right)$$
$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\varphi}$$

$\downarrow$

$$L_+ = \hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\varphi}\right)$$
$$L_- = \hbar e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\cot\theta \frac{\partial}{\partial\varphi}\right)$$

$$L_+ Y_l^l(\theta,\varphi) = \hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right) Y_l^l(\theta,\varphi) = 0$$

$\Rightarrow$ Usando $\underline{Y_l^m}(\theta,\varphi) = F_l^m(\theta)\, e^{im\varphi}$

$$\Rightarrow 0 = \left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)\left[F_l^l(\theta)\, e^{il\varphi}\right] = \left(\frac{\partial}{\partial\theta} - l\cot\theta\right) F_l^l(\theta) = 0$$

$$\Rightarrow \frac{dF_l^l(\theta)}{d\theta} = l\,\frac{\cos\theta}{\operatorname{sen}\theta} F_l^l(\theta) \Rightarrow \frac{1}{F_l^l(\theta)}\frac{dF_l^l(\theta)}{d\theta} = l\,\frac{\cos\theta}{\operatorname{sen}\theta}$$

De ambos lados hay $\frac{g'}{g}$

$$\ln F_l^l(\theta) = l \ln \operatorname{sen}\theta \Rightarrow F_l^l(\theta) = \operatorname{sen}^l(\theta)$$

constante de normalización

$$Y_l^l(\theta,\varphi) = N \sin^l(\theta)\, e^{il\varphi}$$

Normalizamos con

$$\int_0^{2\pi}\int_0^{\pi} |Y_l^l(\theta,\varphi)|^2 \operatorname{sen}\theta\, d\theta\, d\varphi = 1$$

$l=0$	$l=1$	$l=2$
$Y_0^0 = \sqrt{\frac{1}{4\pi}}$	$Y_1^{\pm1} = \pm\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\varphi}$ $Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$	$Y_2^{\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\varphi}$ $Y_2^{\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\varphi}$ $Y_2^0 = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$

La expresión general para cualquier valor de l y m está dada por

$$Y_l^m(\theta,\varphi) = (-1)^m\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\varphi}, \quad m \geq 0;$$
$$Y_l^{-m}(\theta,\varphi) = (-1)^m Y_l^m(\theta,\varphi)^*;$$
$$P_l^m(x) = (1-x^2)^{|m/2|}\frac{d^{|m|}}{dx^{|m|}}P_l(x);$$
$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l.$$

Aquí, a $P_l^m(x)$ se le conoce como polinomio asociado de Legendre.

RELEASE 1.8.0

Python

Search the docs ...

functions (scipy.linalg.lapack)

BLAS Functions for Cython

LAPACK functions for Cython

Interpolative matrix decomposition (scipy.linalg.interpolati

scipy.special.sph_harm

`scipy.special.sph_harm(m, n, theta, phi) = <ufunc 'sph_harm'>`

Compute spherical harmonics.

The spherical harmonics are defined as

$$Y_n^m(\theta,\phi) = \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}e^{im\theta}P_n^m(\cos(\phi))$$

where P_n^m are the associated Legendre functions; see lpmv.

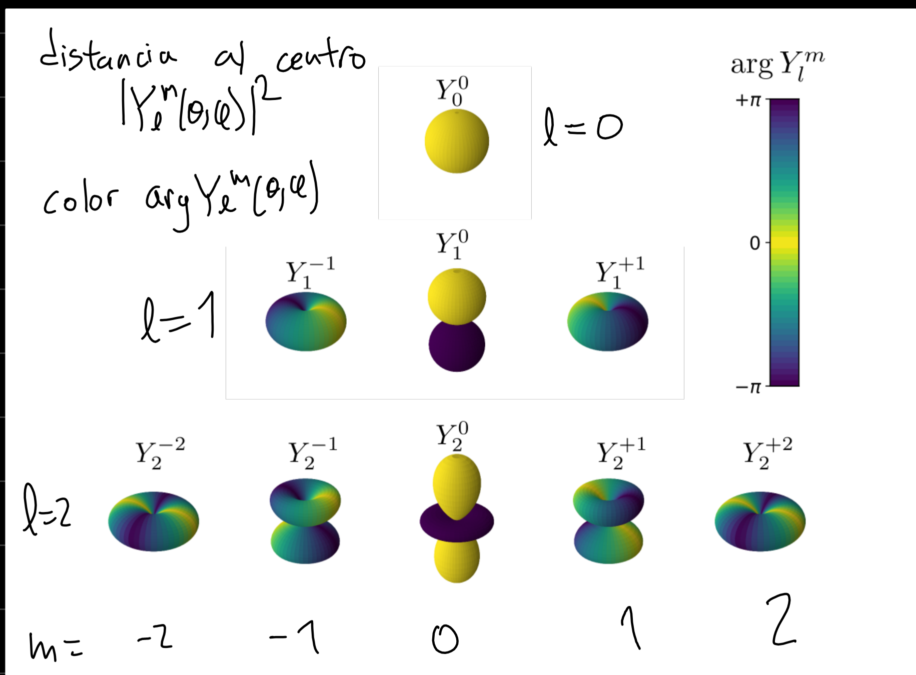
BUILT-IN SYMBOL

Mathematica

SphericalHarmonicY

SphericalHarmonicY[l, m, θ, ϕ]
gives the spherical harmonic $Y_l^m(\theta, \phi)$.

Ojo con la convención usada en cada programa.



- Valores esperados de eigenestados de momento angular.

Considerando $|l,m\rangle$ e-vector de L^2 y L_z

$$\langle L^2\rangle = \langle l,m|L^2|l,m\rangle = \hbar^2 l(l+1)\,\cancel{\langle l,m|l,m\rangle}^{1}$$
$$= \hbar^2 l(l+1)$$

$$\langle L_z\rangle = \hbar m$$

$$\langle L_x\rangle = \langle \tfrac{1}{2}(L_+ + L_-)\rangle = \tfrac{1}{2}\langle l\,m|L_+ + L_-|l\,m\rangle$$

$L_\pm|l,m\rangle = \hbar\sqrt{l(l+1)-m(m\pm1)}\;|l,m\pm1\rangle$

$$= \tfrac{1}{2}\left[\cancel{\langle l\,m|L_+|l\,m\rangle}^{0} + \cancel{\langle l\,m|L_-|l\,m\rangle}^{0}\right]$$
$$= 0$$

$$\langle L_y\rangle = 0$$

$\langle l,m|l,m+2\rangle = 0$ $\langle l,m|l,m-2\rangle$

$$\Delta L_x^2 = \langle L_x^2\rangle = \tfrac{1}{4}\langle l,m|\cancel{L_+^2}^{0} + L_+L_- + L_-L_+ + \cancel{L_-^2}^{0}|l,m\rangle$$

Recordando

$$J^2 = \tfrac{1}{2}(J_+J_- + J_-J_+) + J_z^2 \longrightarrow L_+L_- + L_-L_+ = 2(J^2 - J_z^2)$$

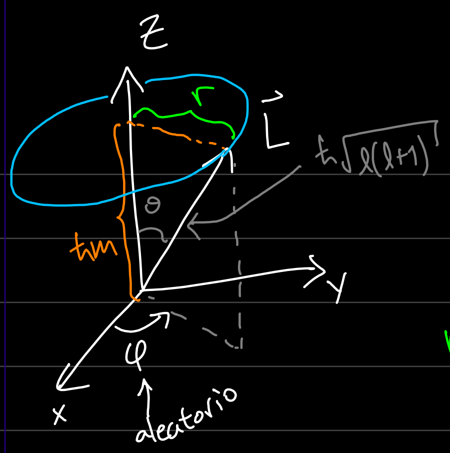
$$\langle L_x^2\rangle = \frac{\hbar^2}{2}\left[l(l+1) - m^2\right]$$

$$\Delta L_y^2 = \langle L_y^2\rangle = \left\langle \left(\frac{L_+ - L_-}{2i}\right)^2\right\rangle = \left\langle -\tfrac{1}{4}(L_+ - L_-)^2\right\rangle$$

$$= -\tfrac{1}{4}\langle L_+^2 - L_+L_- - L_-L_+ + L_-^2\rangle = \cdots = \frac{\hbar^2}{2}\left[l(l+1) - m^2\right]$$

$$\Delta L_x = \Delta L_y = \hbar\sqrt{\tfrac{1}{2}\left[l(l+1) - m^2\right]}$$

$$\Delta L_z^2 = \langle l,m|L_z^2|l,m\rangle - \langle l,m|L_z|l,m\rangle^2 = 0$$

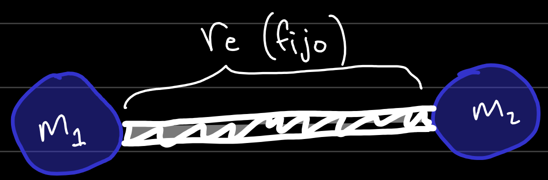


Por teorema de Pitágoras, el radio del círculo

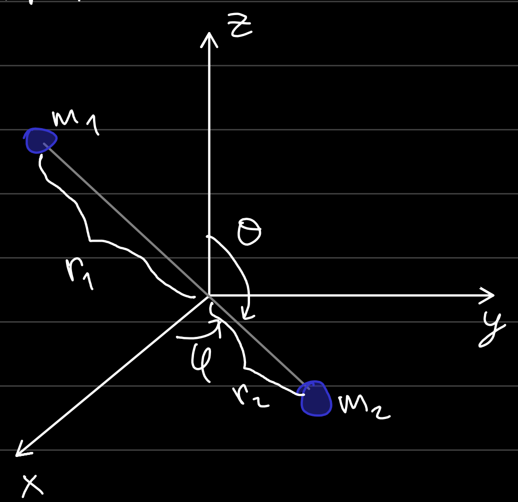
$$r^2 = \hbar^2 l(l+1) - \hbar^2 m^2$$

$$r = \hbar\sqrt{l(l+1) - m^2}$$

Ejemplo rotor rígido.



Tratamiento clasico



- origen en Centro de masa

$$m_1 r_1 = m_2 r_2$$

- Distancia fija $r_1 + r_2 = r_e$
- $I = m_1 r_1^2 + m_2 r_2^2$ momento de inercia
- $\mu = \frac{m_1 m_2}{m_1 + m_2}$ masa reducida

$$I = \mu r_e^2$$

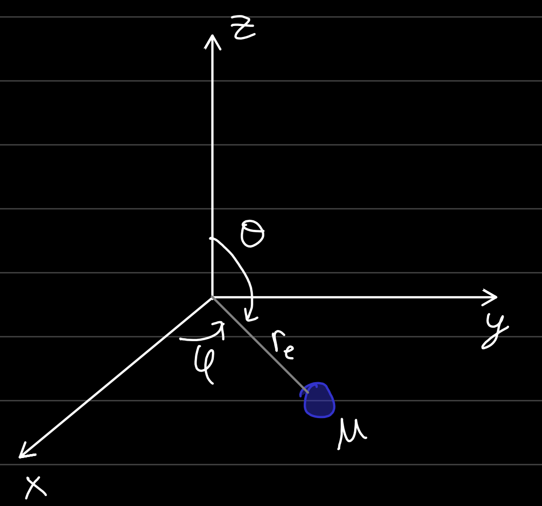
- Si no hay fuerzas externas, el momento angular $\vec{L}$ respecto al C.M. es constante.
  i.e. el rotor gira con frecuencia angular ω_R constante en el plano perpendicular a $\vec{L}$.

$$|\vec{L}| = r_1 m_1 r_1 \omega_R + r_2 m_2 r_2 \omega_R = I \omega_R = \mu r_e^2 \omega_R$$

La energía es la cinética rotacional.

$$H = \frac{1}{2} m \omega_R^2 = \frac{\mathcal{L}^2}{2I} = \frac{\mathcal{L}^2}{2\mu r_e^2}$$

Podemos ver al sistama como una partícula fictica de masa μ separada r_e del origen y que gira con vel. angular ω_R.
$\mathcal{L}$ es el momento angular de la partícula.



Caso cuántico.

El estado del rotor está descrito por

$\Psi(\theta, \varphi)$ (r no importa porque r_e es fijo)

$|\Psi(\theta,\varphi)|^2 \operatorname{sen}\theta \, d\theta \, d\varphi$ es la probabilidad de encontrar al rotor apuntando hacia θ, φ.

$$H = \frac{L^2}{2\mu r_e}$$

Encontrar los e-valores y e-vectores de H se reduce a encontrar los de L^2

$$\Rightarrow \quad H|l,m\rangle = \frac{\hbar^2\, l(l+1)}{2\mu r_e}\,|l,m\rangle$$

E_{lm}

$$\psi_l^m(\theta,\varphi) = Y_l^m(\theta,\varphi)$$

Cada energía es $2l+1$ degenerada

Dado l, m toma los valores

$$m = -l, -l+1, \dots, l-1, l$$