Teoría de perturbaciones II: ejemplos y caso de edos. degenerados

$$H = H_0 + W = H_0 + \lambda \hat{W} \qquad (\lambda \ll 1)$$

conocido — perturbación — parámetro real

$$H_0 |\varphi_p^i\rangle = E_p^0 |\varphi_p^i\rangle$$

degeneración

Buscamos

$$H(\lambda)|\psi(\lambda)\rangle = E(\lambda)|\psi(\lambda)\rangle$$

$$E(\lambda) = \varepsilon_0 + \lambda \varepsilon_1 + \lambda^2 \varepsilon_2 + \dots$$

$$|\psi(\lambda)\rangle = |\psi_0\rangle + \lambda |\psi_1\rangle + \dots$$

Si $|\varphi_n\rangle$ no es degenerado

$$H_0 |\varphi_n\rangle = E_n^0 |\varphi_n\rangle$$

$$E(\lambda) = \underbrace{E_n^0}_{\text{orden} \to 0} + \underbrace{\langle \varphi_n | \lambda \hat{W} | \varphi_n \rangle}_{1} + \underbrace{\sum_p \sum_i \frac{|\langle \varphi_p^i | \lambda \hat{W} | \varphi_n \rangle|^2}{E_n^0 - E_p^0}}_{2}$$

$$|\psi(\lambda)\rangle = \underbrace{|\varphi_n\rangle}_{\text{orden} \to 0} + \underbrace{\sum_p \sum_i \frac{\langle \varphi_p^i | \lambda \hat{W} | \varphi_n \rangle}{E_n^0 - E_p^0} |\varphi_p^i\rangle}_{1}$$

Ejemplo: oscilador armónico cargado en un campo $\vec{\mathcal{E}}$

$$H_0 = \frac{P^2}{2m} + \frac{1}{2} m\omega^2 X^2$$

$$H(\mathcal{E}) = H_0 - q\mathcal{E}X$$

Primero lo resolvemos de manera exacta:

Escribiendo $H(\mathcal{E})|\varphi'\rangle = E'|\varphi'\rangle$ en la base $\{|x\rangle\}$

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2 - q\mathcal{E}x\right]\varphi'(x) = E'\varphi(x)$$

no es ' de derivada

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2\left(x - \frac{q\mathcal{E}}{m\omega^2}\right)^2 - \frac{q^2\mathcal{E}^2}{2m\omega^2}\right]\varphi'(x) = E'\varphi'(x)$$

$$u = x - \frac{q\mathcal{E}}{m\omega^2} \qquad E'' = E' + \frac{q^2\mathcal{E}^2}{2m\omega^2}$$

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{du^2} + \frac{1}{2}m\omega^2u^2\right]\varphi'(u) = E''\varphi'(u)$$

Esta ya la conocemos: ✓

$$E'' = \hbar\omega\left(n + \frac{1}{2}\right) \Rightarrow E'_n = \hbar\omega\left(n + \frac{1}{2}\right) - \frac{q^2\mathcal{E}^2}{2m\omega^2}$$

$$\varphi'_n(u) = \varphi_n\left(x - \frac{q\mathcal{E}}{m\omega^2}\right)$$

eigenfunción de o.a. normalito.

Taylor

$$= \varphi_n(x) - \frac{q\mathcal{E}}{m\omega^2}\frac{d\varphi_n(x)}{dx} + \frac{1}{2!}\left(\frac{q\mathcal{E}}{m\omega^2}\right)^2 \frac{d^2\varphi_n(x)}{dx^2} + \cdots$$

$$= \left(1 - \frac{q\mathcal{E}}{m\omega^2}\frac{d}{dx} + \frac{1}{2!}\left(\frac{q\mathcal{E}}{m\omega^2}\right)^2 \frac{d^2}{dx^2} + \cdots\right)\varphi_n(x)$$

$$= \left[\sum_s \frac{1}{s!}\left(-\frac{q\mathcal{E}}{m\omega^2}\frac{d}{dx}\right)^s\right]\varphi_n(x)$$

$$= e^{-\frac{q\mathcal{E}}{m\omega^2}\frac{d}{dx}}\varphi_n(x)$$

$P = -i\hbar\frac{d}{dx}$

$$= e^{-i\frac{q\mathcal{E}}{\hbar m\omega^2}P}\varphi_n(x)$$

$$|\varphi_n'\rangle = e^{-i\frac{q\mathcal{E}}{\hbar m\omega^2}P}|\varphi_n\rangle \qquad P = \sqrt{m\hbar\omega}\,\hat{P}$$

$$\hat{P} = \frac{i}{\sqrt{2}}(a^+ - a)$$

Recordatorio

$a^+|\varphi_n\rangle = \sqrt{n+1}\,|\varphi_{n+1}\rangle$

$a|\varphi_n\rangle = \sqrt{n}\,|\varphi_{n-1}\rangle$

$$= e^{-\frac{\lambda}{\sqrt{2}}(a^+ - a)}|\varphi_n\rangle$$

con: $\lambda\hbar\omega = -q\mathcal{E}\sqrt{\frac{\hbar}{m\omega}}$

$$\approx \left[1 - \frac{\lambda}{\sqrt{2}}(a^+ - a)\right]|\varphi_n\rangle$$

$$= |\varphi_n\rangle - \lambda\sqrt{\frac{n+1}{2}}\,|\varphi_{n+1}\rangle + \lambda\sqrt{\frac{n}{2}}\,|\varphi_{n-1}\rangle$$

Ahora, usando teoría de perturbaciones:

$$H_0 = \frac{P^2}{2m} + \frac{1}{2}m\omega^2X^2 \qquad E_n^0 = \hbar\omega\left(n+\tfrac{1}{2}\right) \qquad |\varphi_n\rangle$$

eigenvalores — eigenvectores

$$H = H_0 + W \ ; \ W = -q\mathcal{E}X = -q\mathcal{E}\sqrt{\frac{\hbar}{m\omega}}\,\hat{X} = \lambda\hbar\omega\,\hat{X}$$

$$= \frac{\lambda\hbar\omega}{\sqrt{2}}(a^+ + a)$$

Corrección a la energía (1er orden)

$$E_n = E_n^0 + \langle \varphi_n | W | \varphi_n \rangle + O(\lambda^2)$$

$$= \hbar\omega\left(n+\tfrac{1}{2}\right) + \frac{\lambda\hbar\omega}{\sqrt{2}} \cancelto{0}{\underbrace{\langle \varphi_n | a^+ + a | \varphi_n \rangle}_{\xi\langle \varphi_n | \varphi_{n+1}\rangle + \xi\langle \varphi_n | \varphi_{n-1}\rangle}}$$

no hay corrección a primer orden

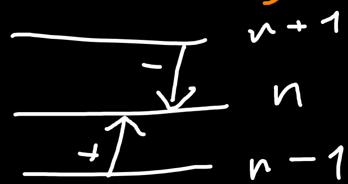
Corrección a segundo orden:

$$E_n = E_n^0 + \cancelto{0}{\langle \varphi_n | W | \varphi_n \rangle} + \sum_{n' \neq n} \frac{|\langle \varphi_{n'} | W | \varphi_n \rangle|^2}{E_n^0 - E_{n'}^0}$$

fijo

Ojo: $\langle \varphi_{n'} | W | \varphi_n \rangle \sim \langle \varphi_{n'} | a^+ + a | \varphi_n \rangle \sim \delta_{n',n+1} + \delta_{n',n-1}$

(sólo quedan 2 términos en la suma)



$$E_n^0 - E_{n'}^0 = \hbar\omega(n - n')$$

$$\langle \varphi_{n+1} | W | \varphi_n \rangle = \lambda\hbar\omega\sqrt{\frac{n+1}{2}}$$

$$\langle \varphi_{n-1} | W | \varphi_n \rangle = \lambda\hbar\omega\sqrt{\frac{n}{2}}$$

$$E_n = \hbar\omega\left(n+\tfrac{1}{2}\right) + 0 - \frac{\lambda^2(\cancel{n}+1)}{2}\hbar\omega + \frac{\lambda^2 \cancel{n}}{2}\hbar\omega$$

$$= \hbar\omega\left(n+\tfrac{1}{2}\right) - \frac{\lambda^2}{2}\hbar\omega$$

$$= \hbar\omega\left(n+\tfrac{1}{2}\right) - \frac{q^2\mathcal{E}^2}{2m\omega^2}$$

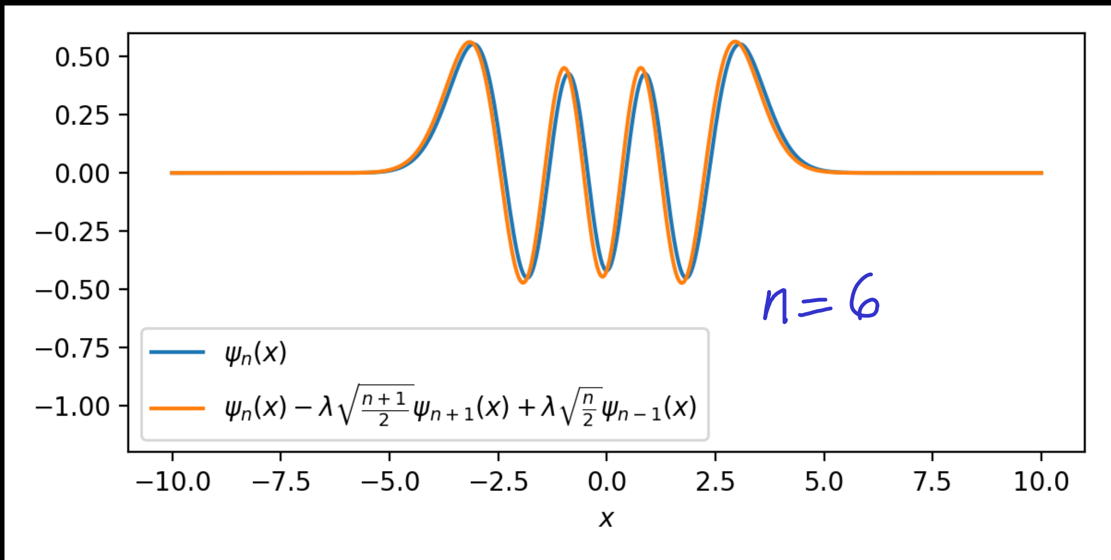
Sol exacta:

$$E_n' = \hbar\omega\left(n+\tfrac{1}{2}\right) - \frac{q^2\mathcal{E}^2}{2m\omega^2}$$

Corrección al eigenvector

$$|\varphi_n'\rangle = |\varphi_n\rangle + \sum_{n'\neq n} \frac{\langle\varphi_{n'}|W|\varphi_n\rangle}{E_n^0 - E_{n'}^0}|\varphi_{n'}\rangle + O(\lambda^2)$$

$$= |\varphi_n\rangle - \lambda\sqrt{\frac{n+1}{2}}\,|\varphi_{n+1}\rangle + \lambda\sqrt{\frac{n}{2}}\,|\varphi_{n-1}\rangle$$



- Ejemplo 2:

Oscilador armónico con perturbación cuadrática

$\rho \ll 1$

$$W = \frac{1}{2}\rho m \omega^2 X^2$$

Solución exacta

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2(1+\rho)X^2$$

Frecuencia de oscilación

$$\omega'^2 = \omega^2(1+\rho)$$

$$E_n = \left(n+\frac{1}{2}\right)\hbar\omega' = \left(n+\frac{1}{2}\right)\hbar\omega\sqrt{1+\rho}$$

$$= \left(n+\frac{1}{2}\right)\hbar\omega\left(1+\frac{\rho}{2}-\frac{\rho^2}{8}+\dots\right)$$

Usando teoría de perturbaciones

$$W = \frac{1}{2}\rho m\omega^2 X^2 = \frac{1}{2}\rho\hbar\omega\hat{X}^2 = \frac{1}{4}\rho\hbar\omega(a^+ + a)^2$$

$$\hat{X} = \frac{1}{\sqrt{2}}(a^+ + a)$$

$$W = \frac{1}{4}\rho\hbar\omega(a^{+2} + a^+a + aa^+ + a^2) = \frac{1}{4}\rho\hbar\omega(a^{+2} + a^2 + 2a^+a + 1)$$

$$E_n = E_0^n + \langle\varphi_n|W|\varphi_n\rangle + \sum_{n'\neq n}\frac{|\langle\varphi_{n'}|W|\varphi_n\rangle|^2}{E_n^0 - E_{n'}^0}$$

$$\hookrightarrow \hbar\omega\left(n+\frac{1}{2}\right)$$

$$\langle\varphi_n|W|\varphi_n\rangle = \langle\varphi_n|\frac{1}{4}\rho\hbar\omega(1+2\overset{N}{\overbrace{a^+a}})|\varphi_n\rangle$$

$$= \frac{1}{4}\rho\hbar\omega(1+2n) = \frac{1}{2}\rho\hbar\omega\left(n+\frac{1}{2}\right)$$

tenemos términos tipo:

$$\langle\varphi_{n'}|a^{+2}|\varphi_n\rangle \sim \delta_{n',n+2}$$

$$\langle\varphi_{n'}|a^2|\varphi_n\rangle \sim \delta_{n',n-2}$$

$$\langle\varphi_{n'}|a^+a|\varphi_n\rangle \sim \delta_{n',n}$$

$$\langle\varphi_{n'}|1|\varphi_n\rangle \sim \delta_{n',n}$$

$$E_n = (n+1)\hbar\omega\left[1+\frac{g}{2}-\frac{g}{8}+\cdots\right] \quad (\text{tarea?})$$

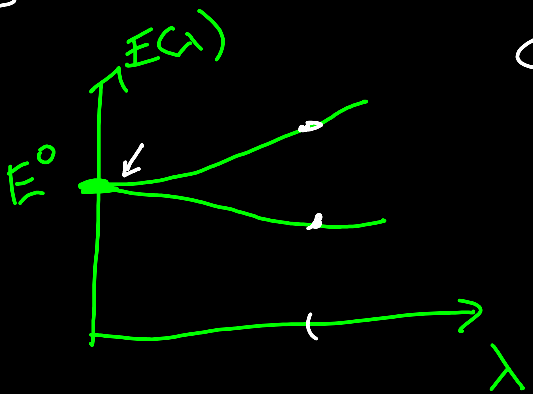
Teoría de perturbaciones para estados degenerados:

- Pensemos en un par de estados degenerados

$$|\psi_a^0\rangle,\ |\psi_b^0\rangle$$

$$H_0|\psi_a^0\rangle = E^0|\psi_a^0\rangle$$

$$H_0|\psi_b^0\rangle = E^0|\psi_b^0\rangle$$



cada línea corresponde a un e.V.

$$\underbrace{|\psi^0\rangle = \alpha|\psi_a^0\rangle + \beta|\psi_b^0\rangle} \text{ es e.V.}$$

¿Cuáles son las combinaciones lineales que corresponden a cada línea?

$$E = \varepsilon_0 + \lambda\varepsilon_1 + \cdots = \sum_q \varepsilon_q \lambda^q$$

$$|\psi\rangle = |\psi_0\rangle + \lambda|\psi_1\rangle + \cdots = \sum_q \lambda^q|\psi_q\rangle$$

$$H|\psi\rangle = E|\psi\rangle$$

$$(H+\underset{\uparrow \lambda\hat{W}}{W})\left(\sum_q \lambda^q|\psi_q\rangle\right) = \left(\sum_{q'} \lambda^{q'}\varepsilon_{q'}\right)\left(\sum_q \lambda^q|\psi_q\rangle\right)$$

Agrupando por potencias de λ:

Primer orden:

$$\hat{W}|\psi_0\rangle + H_0|\psi_1\rangle = \varepsilon_1|\psi_0\rangle + \varepsilon_0|\psi_1\rangle$$

Haremos producto interno con $|\psi_a^0\rangle$ y $|\psi_b^0\rangle$

$$\langle\psi_a^0|\hat{W}|\psi_0\rangle + \langle\psi_a^0|\cancel{H_0|\psi_1\rangle} = \varepsilon_1\langle\psi_a^0|\psi_0\rangle + \varepsilon_0\cancel{\langle\psi_a^0|\psi_1\rangle}$$

ε_0

$$\langle\psi_b^0|\hat{W}|\psi_0\rangle + \langle\psi_b^0|\cancel{H_0|\psi_1\rangle} = \varepsilon_1\langle\psi_b^0|\psi_0\rangle + \varepsilon_0\cancel{\langle\psi_b^0|\psi_1\rangle}$$

ε_0

$$H_0|\psi_a^0\rangle = \varepsilon_0|\psi_a^0\rangle$$
$$= E^0|\psi_a^a\rangle$$

$$\langle\psi_a^0|\hat{W}|\psi_0\rangle = \varepsilon_1\langle\psi_a^0|\psi_0\rangle$$

$$\langle\psi_b^0|\hat{W}|\psi_0\rangle = \varepsilon_1\langle\psi_b^0|\psi_0\rangle$$

Si $|\psi_0\rangle = \alpha|\psi_a^0\rangle + \beta|\psi_b^0\rangle$

$$\alpha\langle\psi_a^0|\hat{W}|\psi_a^0\rangle + \beta\langle\psi_a^0|\hat{W}|\psi_b^0\rangle = \varepsilon_1\alpha$$

$$\alpha\langle\psi_b^0|\hat{W}|\psi_a^0\rangle + \beta\langle\psi_b^0|\hat{W}|\psi_b^0\rangle = \varepsilon_1\beta$$

$$\begin{pmatrix} W_{aa} & W_{ab} \\ W_{ba} & W_{bb} \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \varepsilon_1 \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Lo que no conocemos es

α, β, ε_1

$$W_{ij} = \langle \psi_i^0 | \hat{W} | \psi_j^0 \rangle \qquad i,j \in \{a,b\}$$

Moraleja:

- 1: Escribir W como matriz en la base de e.V. degenerados

- 2: Al encontrar los e.v. de esta matriz encontramos ε_1.